The alternative model of the spherical oscillator

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Abstract

The quasiradial wave functions and energy spectra of the alternative model of spherical oscillator on the D-dimensional sphere and two-sheeted hyperboloid are found.

Keywords: Spherical oscillator, sphere, two-sheeted hyperboloid.

1 Introduction

The spherical oscillator was suggested by Higgs [1, 2]. The D-dimensional spherical oscillator is defined by the potential

$$V_{S^D} = \frac{\omega^2}{2}\frac{x_\mu x_\mu}{x_0^2}, \qquad \mu = 1, 2, \ldots, D, \tag{1}$$

where $x_0,\ x_\mu$ are the Euclidean coordinates of the ambient space $\mathbb{R}^{D+1}$: $x_0^2+x_\mu x_\mu = r_0^2$ for D-dimensional sphere and $x_0^2 - x_\mu x_\mu = r_0^2$ for D-dimensional two-sheeted hyperboloid. (We use a system of units in which the reduced mass m and Planck constant $\hbar$ satisfy $m = \hbar = 1$.) The spherical oscillator (1) on the D-dimensional sphere and two-sheeted hyperboloid is considered in [3] in detail.

The oscillator problem on spheres and pseudospheres was discussed from many point of view in [4, 5, 6, 7, 8, 9, 10].

The alternative model of spherical oscillator, which was suggested in our previous papers [11, 12], is defined by the potential

$$V_S^D = 2\omega^2 r_0^2 \frac{r_0 - x_0}{r_0 + x_0} \tag{2}$$

on the D-dimensional sphere, and

$$V_H^D = 2\omega^2 r_0^2 \frac{x_0 - r_0}{x_0 + r_0} \tag{3}$$

on the D-dimensional two-sheeted hyperboloid.

The two-dimensional case of the oscillator potentials (2) and (3) was considered in [13, 14].

2 Quasiradial function on D-sphere

The Schrödinger equation describing the nonrelativistic quantum motion in the D-dimensional curved space has the following form:

$$\hat{H}\Psi = \left[-\frac{1}{2}\Delta_{LB} + V\left(\vec{x}\right)\right]\Psi = E\Psi, \tag{4}$$

where the Laplace-Beltrami operator in arbitrary curvilinear coordinates ξ_μ is

$$\Delta_{LB} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi_\mu}\left(g^{\mu\nu}\sqrt{g}\frac{\partial}{\partial\xi_\mu}\right), \qquad g = det g_{\mu\nu}, \qquad g_{\alpha\mu}g^{\mu\beta} = \delta_\alpha^\beta.$$

In the hyperspherical coordinates

$$\begin{aligned}
x_0 &= r_0\cos\chi,\\
x_1 &= r_0\sin\chi\cos\theta_1,\\
x_2 &= r_0\sin\chi\sin\theta_1\cos\theta_2,\\
&\vdots\\
x_{D-1} &= r_0\sin\chi\sin\theta_1\sin\theta_2\cdots\sin\theta_{D-2}\cos\varphi,\\
x_D &= r_0\sin\chi\sin\theta_1\sin\theta_2\cdots\sin\theta_{D-2}\sin\varphi,
\end{aligned}$$

where $\chi, \theta_1, \ldots, \theta_{D-2} \in [0, \pi]$, $\varphi \in [0, 2\pi)$, the oscillator potential (2) reads

$$V_S^D = 2\omega^2 r_0^2 \tan^2 \frac{\chi}{2}. \tag{5}$$

The Schrödinger equation (4) for the potential (5) may be solved by searching for a wave function in the form

$$\Psi\left(\chi, \theta_1, \ldots, \theta_{D-2}, \varphi\right) = R(\chi)\, Y_{Ll_1 l_2 \ldots l_{D-2}}\left(\theta_1, \ldots, \theta_{D-2}, \varphi\right),$$

where l_i are the angular hypermomenta and L is total angular momentum, and the hyperspherical function $Y_{Ll_1 l_2 \ldots l_{D-2}}\left(\theta_1, \ldots, \theta_{D-2}, \varphi\right)$ is the solution of the Laplace-Beltrami eigenvalue equation on the (D-1)-dimensional sphere. After the separation of variables in (4) we obtain the quasiradial equation

$$\frac{1}{(\sin\chi)^{D-1}} \frac{\partial}{\partial\chi}\left[(\sin\chi)^{D-1}\frac{\partial R}{\partial\chi}\right] + \left[2r_0^2 E - \frac{L(L+D-2)}{\sin^2\chi} - 4\omega^2 r_0^4 \tan^2\frac{\chi}{2}\right] R = 0.$$

Using the substitution

$$R(\chi) = (\sin\chi)^{-\frac{D-1}{2}}\, Z(\chi)$$

we find the Pöschl-Teller type equation

$$\frac{d^2 Z}{d\xi^2} + \left[\epsilon - \frac{\nu^2 - \frac{1}{4}}{\cos^2\xi} - \frac{\left(L + \frac{D-2}{2}\right)^2 - \frac{1}{4}}{\sin^2\xi}\right] Z = 0, \tag{6}$$

where $\xi = \frac{\chi}{2} \in \left[0, \frac{\pi}{2}\right]$, and

$$\epsilon = 8r_0^2 E + (D-1)^2 + 16\omega^2 r_0^4, \qquad \nu = \sqrt{\left(L + \frac{D-2}{2}\right)^2 + 16\omega^2 r_0^4}.$$

The solution of Eq. (6) regular for $\xi \in \left[0, \frac{\pi}{2}\right]$ and expressed in terms of the hypergeometric function is [15]

$$R_{n_r L\nu}^D(\chi) = C_{n_r L\nu}^D \left(\sin\frac{\chi}{2}\right)^L \left(\cos\frac{\chi}{2}\right)^{\nu - \frac{D}{2} + 1} \times \tag{7}$$

$$\times {}_2F_1\left(-n_r,\, n_r + L + \nu + \frac{D}{2};\, L + \frac{D}{2};\, \sin^2\frac{\chi}{2}\right),$$

and the ϵ is quantized as

$$\epsilon = \left(2n_r + L + \nu + \frac{D}{2}\right)^2,$$

where $n_r = 0, 1, 2, \ldots$ is a "quasiradial" quantum number. The eigenvalues E are given by

$$E_N^D = \frac{1}{8r_0^2}\left[(N+1)(N+D) + (2\nu - 1)\left(N + \frac{D}{2}\right) + L(L+D-2) - \frac{D}{2}(D-1)\right], \tag{8}$$

where $N = 2n_r + L = 0, 1, 2, \ldots$ is the principal quantum number.

For the quasiradial wave function $R_{n_r L\nu}^D(\chi)$ we choose the normalization condition

$$r_0^D \int_0^\pi \left|R_{n_r L\nu}^D(\chi)\right|^2 (\sin\chi)^{D-1}\, d\chi = 1$$

and find:

$$C_{n_r L\nu}^D = \sqrt{\frac{\left(2n_r + L + \nu + \frac{D}{2}\right)\Gamma\left(n_r + L + \nu + \frac{D}{2}\right)\Gamma\left(n_r + L + \frac{D}{2}\right)}{2^{D-1}\, r_0^D\, (n_r)!\, \Gamma\left(n_r + \nu + 1\right)\left[\Gamma\left(L + \frac{D}{2}\right)\right]^2}}. \tag{9}$$

In the limit $r_0 \to \infty$, $\chi \to 0$ and $\chi r_0 \sim r$ - fixed and $\nu \sim 4\omega r_0^2$, we see that

$$\lim_{r_0\to\infty} E_N^D = \omega\left(N + \frac{D}{2}\right) \tag{10}$$

and

$$\lim_{r_0\to\infty} R_{NL\nu}^D(\chi) = \frac{\omega^{\frac{L}{2}+\frac{D}{4}}}{\Gamma\left(L+\frac{D}{2}\right)}\sqrt{\frac{2\Gamma\left(\frac{N+L+D}{2}\right)}{\left(\frac{N-L}{2}\right)!}}\, r^L\, e^{-\frac{\omega r^2}{2}}\, F\left(-\frac{N-L}{2}; L+\frac{D}{2}; \omega r^2\right), \tag{11}$$

where $F(a;c;x)$ is the confluent hypergeometric function. Formula (11) coincides with the known formula for D-dimensional flat radial wave functions [16].

3 Oscillator on the D-dimensional hyperboloid

The pseudospherical coordinates on the D-dimensional two-sheeted hyperboloid: $x_0^2 - x_1^2 - x_2^2 - x_D^2 = r_0^2$, $x_0 \geq r_0$, are

$$\begin{aligned}
x_0 &= r_0 \cosh\tau,\\
x_1 &= r_0 \sinh\tau \cos\theta_1,\\
x_2 &= r_0 \sinh\tau \sin\theta_1 \cos\theta_2,\\
&\vdots\\
x_{D-1} &= r_0 \sinh\tau \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{D-2} \cos\varphi,\\
x_D &= r_0 \sinh\tau \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{D-2} \sin\varphi,
\end{aligned}$$

where $\tau \in [0,\infty)$. Variables in the Schrödinger equation (4) may be separated for oscillator potential (3) which in the pseudospherical coordinates has the form

$$V_H^D = 2\omega^2 r_0^2 \tanh^2\frac{\tau}{2},$$

by the ansatz

$$\Psi(\tau, \theta_1, \ldots, \theta_{D-2}, \varphi) = R(\tau)\, Y_{Ll_1l_2\ldots l_{D-2}}(\theta_1, \ldots, \theta_{D-2}, \varphi),$$

where, as in the previous case l_i, are the angular hypermomenta and L is the total angular momentum, and the hyperspherical function $Y_{Ll_1l_2\ldots l_{D-2}}(\theta_1, \ldots, \theta_{D-2}, \varphi)$ is the solution of the Laplace-Beltrami eigenvalue equation on the (D-1)-dimensional sphere. After separation of variables in (4) we find the quasiradial equation

$$\frac{1}{(\sinh\tau)^{D-1}}\frac{\partial}{\partial\tau}\left[(\sinh\tau)^{D-1}\frac{\partial R}{\partial\tau}\right] + \left[2r_0^2E - \frac{L(L+D-2)}{\sinh^2\tau} - 4\omega^2 r_0^4\tanh^2\frac{\tau}{2}\right]R = 0.$$

Using now the substitution

$$R(\tau) = (\sinh\tau)^{-\frac{D-1}{2}}\, Z(\tau)$$

we come to the equation

$$\frac{d^2Z}{d\rho^2} + \left[\epsilon - \frac{\nu^2 - \frac{1}{4}}{\cosh^2\rho} - \frac{\left(L+\frac{D-2}{2}\right)^2 - \frac{1}{4}}{\sinh^2\rho}\right]Z = 0, \tag{12}$$

where $\rho = \frac{\tau}{2} \in [0,\infty)$, and $\epsilon = 8r_0^2 - (D-1)^2 - 16\omega^2 r_0^4$.

Thus, the oscillator problem on the two-sheeted hyperboloid is described by the modified Pöschl-Teller equation and, unlike the oscillator equation on the sphere which has only a discrete spectrum, equation (12) possesses both bound and unbound states.

The discrete quasiradial wave function regular on the line $\tau \in [0, \infty)$ and normalized by the condition

$$r_0^D \int_0^\infty \left| R_{n_r L \nu}^D(\tau) \right|^2 (\sinh \tau)^{D-1} \, d\tau = 1$$

has the form

$$R_{n_r L \nu}^D(\tau) = \frac{1}{\Gamma\left(L + \frac{D}{2}\right)} \sqrt{\frac{\left(\nu - 2n_r - L - \frac{D}{2}\right) \Gamma\left(\nu - n_r\right) \Gamma\left(n_r + L + \frac{D}{2}\right)}{2^{D-1} r_0^D \, (n_r)! \, \Gamma\left(\nu - n_r - L - \frac{D}{2} + 1\right)}} \times \tag{13}$$
$$\times \left(\sinh \frac{\tau}{2}\right)^L \left(\cosh \frac{\tau}{2}\right)^{2n_r - \nu - \frac{D}{2} + 1} \times {}_2F_1\left(-n_r, \, -n_r + \nu; \, L + \frac{D}{2}; \, \tanh^2 \frac{\tau}{2}\right),$$

with the "quasiradial" quantum number $n_r = 0, 1, 2, \ldots, \left[\frac{1}{2}\left(\nu - L - \frac{D}{2}\right)\right]$. The ϵ is quantized by

$$\epsilon = -\left(2n_r + L - \nu + \frac{D}{2}\right)^2,$$

and the energy spectrum for the alternative model of quantum spherical oscillator on the D-dimensional two-sheeted hyperboloid takes the value

$$E_N^D = \frac{1}{8r_0^2} \left[(2\nu - 1)\left(N + \frac{D}{2}\right) - N(N + D - 1) - L(L + D - 2) + \frac{D}{2}(D - 1)\right]. \tag{14}$$

Here $N = 2n_r + L$ is the principal quantum number and the bound state solution is possible only for

$$0 \le N \le \left[\nu - \frac{D}{2}\right].$$

In the contractio limit $r_0 \to \infty$, $\tau \sim r/r_0$ and $\nu \sim 4\omega r_0^2$, we see that the continuous spectrum vanishes while the discrete spectrum is infinite, and it is easy to reproduce the oscillator energy spectrum (10) and wave function (11).

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References

[1] P.W. Higgs: *J. Phys.* **A12**, 309–323 (1979).

[2] H.I. Leemon: *J. Phys.* **A12**, 489–501 (1979).

[3] E.G. Kalnins, W.J. Miller, G.S. Pogosyan: *Phys. At. Nucl.* **65**, 1086–1094 (2002).

[4] D. Bonatos, C. Daskaloyannis, K. Kokkotas: *Phys. Rev.* **A50**, 3700–3709 (1994).

[5] C. Grocshe, G.S. Pogosyan, A.N. Sissakian: *Fortschritte der Physik* **43**, 523–563 (1995).

[6] C. Grocshe, G.S. Pogosyan, A.N. Sissakian: *Phys. Part. Nucl.* **27**, 244–278 (1996).

[7] C. Grocshe, G.S. Pogosyan, A.N. Sissakian: *Phys. Part. Nucl.* **28**, 486–519 (1997).

[8] E.G. Kalnins, W.J. Miller, G.S. Pogosyan: *J. Mat. Phys.* **37**, 6439–6467 (1996).

[9] E.G. Kalnins, W.J. Miller, G.S. Pogosyan: *J. Mat. Phys.* **38**, 5416–5433 (1997).

[10] L. Mardoyan, A. Nersessian: *Phys. Rev.* **B72**, 233303 (2005).

[11] S. Bellucci, L. Mardoyan, A. Nersessian: *Phys. Lett.* **B636**, 137–141 (2006).

[12] L. Mardoyan, A. Nersessian, A Yeranyan: *Phys. Lett.* **A366**, 30–35 (2007).

[13] S. Bellucci, A. Nersessian: *Phys. Rev.* **D67**, 065013 (2003).

[14] S. Bellucci, A. Nersessian, A. Yeranyan: *Phys. Rev.* **D70**, 085013 (2004).

[15] S. Flügge: *Problems in quantum mechanics*, V.1, Springer-Verlag, Berlin-Heidelberg-New York 1971.

[16] L.G. Mardoyan, G.S. Pogosyan, A.N. Sissakian, V.M. Ter-Antonyan: *Quantum systems with hidden symmetry*, FIZMATLIT Publ., Moscow 2006 (in Russian).